Rare-event simulation: Code demo 2

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1 Importance sampling

1.1 Setup

[1]:
```
# numpy is the 'Numerical Python' package
import numpy as np

# Numpy's methods for pseudorandom number generation
import numpy.random as rnd

# Import the plotting library matplotlib
import matplotlib.pyplot as plt
```

[2]:
```
# Print out the versions of software I'm running
import sys
print("Python version:", sys.version)
print("Numpy version:", np.__version__)
```

```
Python version: 3.7.6 | packaged by conda-forge | (default, Jan  7 2020,
21:00:34) [MSC v.1916 64 bit (AMD64)]
Numpy version: 1.17.4
```

[3]:
```
# Reminder that we need a relatively new version of numpy to make
# use of the latest pseudorandom number generation algorithms.
if int(np.__version__.split('.')[1]) < 17:
    raise RuntimeError("Need Numpy version >= 1.17")
```

Let's try to approximate some tail probability for a normal distribution. E.g. take $X \sim \mathsf{Normal}(1, 2^2)$, and try to estimate $\mathbb{P}(X > \gamma)$.

Frankly, we don't need to approximate this, since we have

$$\mathbb{P}(X > \gamma) = \mathbb{P}(2Z + 1 > \gamma) = \mathbb{P}\big(Z > \frac{\gamma - 1}{2}\big) = \Phi\big(-\frac{\gamma - 1}{2}\big)$$

where $Z \sim \mathsf{Normal}(0, 1)$ and Φ is the standard normal c.d.f.

But let's pretend we couldn't calculate this, and needed to use crude Monte Carlo (CMC) to approximate it. The CMC approximation involve sampling a large number of i.i.d. X's and looking at the fraction of these which are greater than γ. Let's start with $\gamma = 5$.

[4]:
```
# scipy is the 'Scientific Python' package
# We'll use the stats package to get some
# p.d.f.s & c.d.f.s
from scipy import stats

γ = 5
μ = 1
σ = 2 # <-- Note, not σ^2!

R = 10**4
rng = rnd.default_rng(1)
normals = rng.normal(μ, σ, R)

ests = normals > γ
ellHat = ests.mean()
sigmaHat = ests.std()
widthCI = 1.96 * sigmaHat / np.sqrt(R)

print(f"CMC estimate:\t {ellHat} (+/- {widthCI})")
print(f"CMC low bound:\t {ellHat-widthCI}")
print(f"CMC upp bound:\t {ellHat+widthCI}")
print(f"Theoretical:\t {stats.norm.cdf(-(γ-1)/2)}")
```

```
CMC estimate:    0.0216 (+/- 0.0028493196223660132)
CMC low bound:   0.018750680377633990
CMC upp bound:   0.024449319622366013
Theoretical:     0.022750131948179195
```

This seems to work well. How about using MC to estimate $\mathbb{P}(X > 10)$? Using the theory from above we know the real probability is:

[5]:
```
stats.norm.cdf(-(10-1)/2)
```

[5]:
```
3.3976731247300535e-06
```

Yet using CMC gives us the sad answer of

[6]:
```
mcEstimate = np.mean(normals > 10)
print("CMC estimate:", mcEstimate)
```

```
CMC estimate: 0.0
```

What's even worse, is that CMC is very confident about this wrong answer!

[7]:
```
ests = normals > 10
sigmaHat = ests.std()
widthCI = 1.96 * sigmaHat / np.sqrt(R)
print("Confidence interval width:", widthCI)
```

```
Confidence interval width: 0.0
```

We use importance sampling, and sample from a $\mathsf{Normal}(\mu', 2^2)$ distribution (i.e. we shift the mean of the original distribution). Let's go back to $\gamma = 5$ first.

[8]:
```
γ = 5

# Sample from the new distribution
μDash = γ
normals = rng.normal(μDash, σ, R)

# Calculate the likelihood ratios
lrNumer = stats.norm.pdf(normals, μ, σ)
lrDenom = stats.norm.pdf(normals, μDash, σ)
lrs = lrNumer / lrDenom

# Construct estimate and CI's
ests = lrs * (normals > γ)
ellHat = ests.mean()
sigmaHat = ests.std()
widthCI = 1.96 * sigmaHat / np.sqrt(R)
print(f"IS estimate:\t {ellHat} (+/- {widthCI})")
print(f"IS low bound:\t {ellHat-widthCI}")
print(f"IS upp bound:\t {ellHat+widthCI}")
print(f"Theoretical:\t {stats.norm.cdf(-(γ-1)/2)}")
```

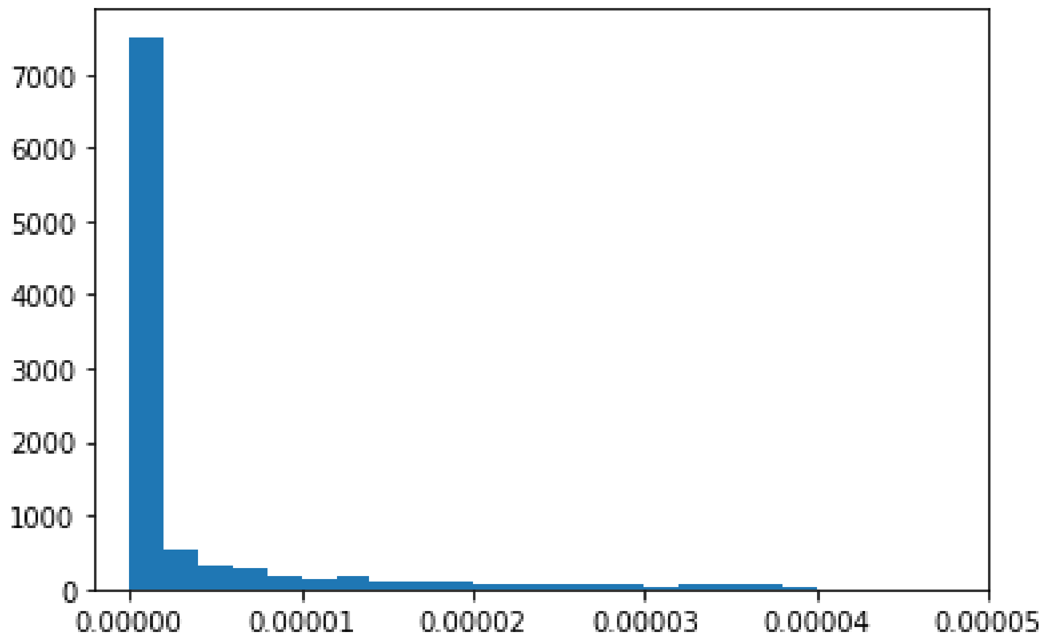
```
IS estimate:     0.023044493206409045 (+/- 0.0006875343072615055)
IS low bound:    0.022356958899147 54
IS upp bound:    0.02373202751367055
Theoretical:     0.022750131948179195
```

[9]:
```
γ = 10

# Sample from the new distribution
μDash = γ
normals = rng.normal(μDash, σ, R)

# Calculate the likelihood ratios
lrNumer = stats.norm.pdf(normals, μ, σ)
lrDenom = stats.norm.pdf(normals, μDash, σ)
lrs = lrNumer / lrDenom

# Construct estimate and CI's
ests = lrs * (normals > γ)
ellHat = ests.mean()
sigmaHat = ests.std()
widthCI = 1.96 * sigmaHat / np.sqrt(R)
print(f"IS estimate:\t {ellHat} (+/- {widthCI})")
```

```
print(f"IS low bound:\t {ellHat-widthCI}")
print(f"IS upp bound:\t {ellHat+widthCI}")
print(f"Theoretical:\t {stats.norm.cdf(-(γ-1)/2)}")
```

```
IS estimate:     3.394413020005718e-06 (+/- 1.4987704226621852e-07)
IS low bound:    3.2445359777394996e-06
IS upp bound:    3.544290062271937e-06
Theoretical:     3.3976731247300535e-06
```

[10]:
```
plt.hist(ests, 20);
plt.xticks([0, 1e-5, 2e-5, 3e-5, 4e-5, 5e-5]);
```

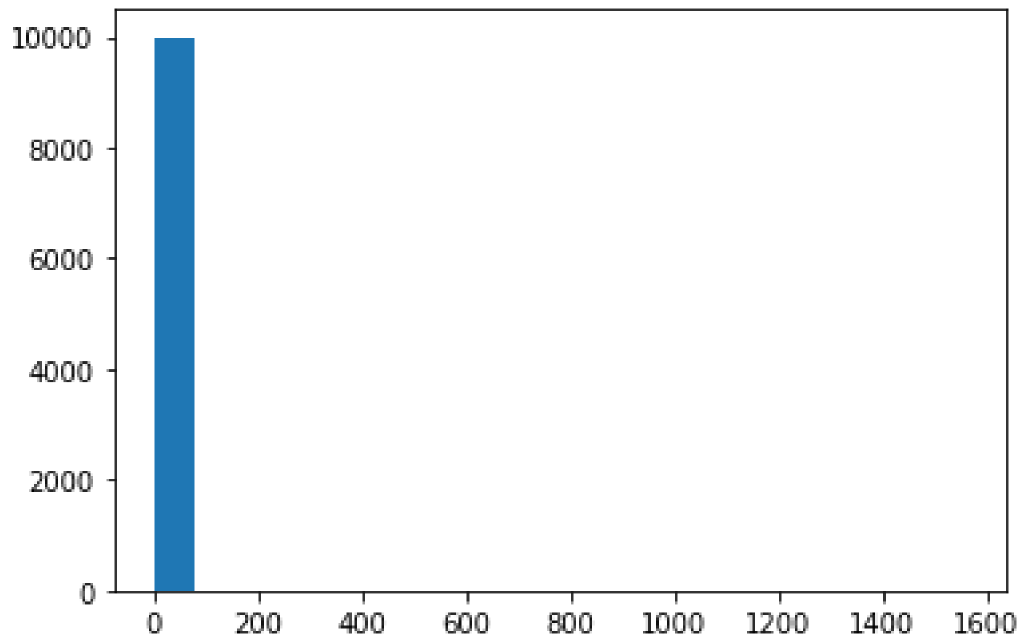


[11]:
```
np.max(ests), np.argmax(ests)
```

[11]:
```
(4.002058099702878e-05, 690)
```

[12]:
```
plt.hist(lrs, 20);
lrs.mean()
```

[12]:
```
0.3394246408190732
```



2 Siegmund's Algorithm

We model an insurer's risk reserve process R_t as

$$R(t) = u + pt - \sum_{i=1}^{N_t} U_i$$

where $u \geq 0$, $p > 0$, N_t is a Poisson process with intensity λ and $U_i \overset{\text{i.i.d.}}{\sim} \mathsf{Exponential}(\lambda_U)$.

For this example, it's easier to work with the net payout

$$P(t) = \sum_{i=1}^{N_t} U_i - pt \,,$$

The only possible times when the insurer's reserve can become negative is at the times $T_1, T_2, \ldots$ when the claims arrive. If we denote the interarrival times as $\xi_i = T_i - T_{i-1} \sim \mathsf{Exponential}(\lambda_\xi)$ (letting $T_0 \equiv 0$), then we have the running sum form

$$S_n := P(T_n) = \sum_{i=1}^{n} X_i \quad \text{where} \quad X_i = U_i - p\xi_i \,.$$

As $V_i := p\xi_i \sim \mathsf{Exponential}(\lambda_V)$ where $\lambda_V = \lambda_\xi \,/\, p$, then we instead use

$$S_n = \sum_{i=1}^{n} X_i \quad \text{where} \quad X_i = U_i - V_i \,.$$

The **time of ruin** given that our initial capital is u is

$$\tau_u = \min\{n \geq 1 : S_n > u\}$$

and so our **infinite time ruin probability** is

$$\mathbb{P}(\tau_u < \infty)\,.$$

We can roughly estimate this with crude Monte Carlo.

[13]:
```
%%time

rng = rnd.default_rng(1)

u = 1
p = 0.5
λ_U = 6
λ_ξ = 0.005
λ_V = λ_ξ / p
giveUpTime = 200
R = 10**6

alive = np.full(R, True)
S_n = np.zeros(R)

for n in range(1, giveUpTime):
    U_n = rng.exponential(1/λ_U, R)
    V_n = rng.exponential(1/λ_V, R)
    X_n = U_n - V_n
    S_n += X_n

    bankruptNow = (S_n > u) & alive

    alive[bankruptNow] = False

    if np.sum(alive) == 0:
        break

ellHat = np.mean(~alive)

print(f"CMC lower bound estimate:\t {ellHat}")
```

```
CMC lower bound estimate:       3e-06
Wall time: 12.4 s
```

Let's exponentially tilt the X_i to make them bigger. Say $X_i \sim f(\cdot)$ and $M(\theta) = \mathbb{E}_f[\mathrm{e}^{\theta X}]$. The proposal distribution is

$$g(x) = \frac{\mathrm{e}^{\theta x}}{M(\theta)} f(x) = \mathrm{e}^{\theta x - \kappa(\theta)} f(x)$$

where $\kappa(\theta) := \log M(\theta)$.

The likelihood ratio for a sequence $X_1, X_2, \dots, X_{\tau_u}$ is

$$L = \prod_{i=1}^{\tau_u} \frac{f(X_i)}{g(X_i)} = \prod_{i=1}^{\tau_u} \frac{f(X_i)}{\mathrm{e}^{\theta X_i - \kappa(\theta)} f(X_i)} = \exp\big\{-\theta S_{\tau_u} + \tau_u \kappa(\theta)\big\} .$$

Thus, our estimate is

$$\mathbb{P}(\tau_u < \infty) \approx \frac{1}{R} \sum_{r=1}^{R} 1\{\tau_u^{(r)} < \infty\} \exp\big\{-\theta S_{\tau_u^{(r)}} + \tau_u^{(r)} \kappa(\theta)\big\} =: \hat{\ell}_{\mathrm{IS}} .$$

Does this make bankruptcy more likely? Let's calculate the mean of the tilted summands:

$$\mathbb{E}_g[X] = \mathbb{E}_f\big[X \frac{g(X)}{f(X)}\big] = \frac{\mathbb{E}_f\big[X \mathrm{e}^{\theta X}\big]}{M(\theta)} .$$

Since

$$M'(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta} \mathbb{E}\big[\mathrm{e}^{\theta X}\big] = \mathbb{E}_f\big[\frac{\mathrm{d}}{\mathrm{d}\theta} \mathrm{e}^{\theta X}\big] = \mathbb{E}_f\big[X \mathrm{e}^{\theta X}\big] ,$$

and as $\kappa'(\theta) = M'(\theta)/M(\theta)$ we conclude

$$\mathbb{E}_g[X] = \kappa'(\theta) .$$

Thus, we should choose θ such that $\mathbb{E}_g[X] \geq 0$ and we will always simulate bankruptcy events $\mathbb{P}_g(\tau_u < \infty) = 1$.

What is the moment generating function $M(\theta)$? Remember $X = U - V$ where

$$U \sim \mathsf{Exponential}(\lambda_U) \text{ and } V \sim \mathsf{Exponential}(\lambda_V) .$$

Also remember $E \sim \mathsf{Exponential}(\lambda)$ has $M_E(\theta) = \lambda/(\lambda - \theta)$ for $\theta < \lambda$.

Then

$$\begin{aligned} M_X(\theta) &= \mathbb{E}\big[\mathrm{e}^{\theta(U-V)}\big] = \mathbb{E}\big[\mathrm{e}^{\theta U}\big]\mathbb{E}\big[\mathrm{e}^{-\theta V}\big] \\ &= M_U(\theta)M_V(-\theta) = \frac{\lambda_U}{\lambda_U-\theta}\frac{\lambda_V}{\lambda_V+\theta}\,. \end{aligned}$$

Which tilting parameter θ do we choose? First requirement is that θ is large enough that $\kappa'(\theta) \geq 0$ so $\mathbb{P}_g(\tau_u < \infty) = 1$. Then

$$\begin{aligned} \hat{\ell}_{\mathrm{IS}} &= \frac{1}{R}\sum_{r=1}^{R} 1\{\tau_u^{(r)} < \infty\}\exp\big\{-\theta S_{\tau_u^{(r)}} + \tau_u^{(r)}\kappa(\theta)\big\} \\ &= \frac{1}{R}\sum_{r=1}^{R}\exp\big\{-\theta S_{\tau_u^{(r)}}\big\} \end{aligned}$$

if $\kappa(\theta) = 0$. This corresponds to solving for γ

$$M_X(\gamma) = \frac{\lambda_U}{\lambda_U-\gamma}\frac{\lambda_V}{\lambda_V+\gamma} = 1$$

Has solution $\gamma = \lambda_U - \lambda_V$ (to check $M_X(\gamma) = \frac{\lambda_U}{\lambda_V}\frac{\lambda_V}{\lambda_U} = 1$).

How do we simulate from this distribution? So we've chosen the proposal distribution for IS to be

$$g(x) = \frac{\mathrm{e}^{\gamma x}}{M_X(\gamma)} f(x) = \mathrm{e}^{\gamma x} f(x)\,.$$

Under the tilted distribution g where we tilt by γ, the X has moment generating function

$$\begin{aligned} \mathbb{E}_g\big[\mathrm{e}^{\theta X}\big] &= \int \mathrm{e}^{\theta x} g(x)\,\mathrm{d}x = \int \mathrm{e}^{\theta x}\mathrm{e}^{\gamma x} f(x)\,\mathrm{d}x \\ &= \int \mathrm{e}^{(\theta+\gamma)x} f(x)\,\mathrm{d}x = M_X(\theta+\gamma) \\ &= \frac{\lambda_U}{\lambda_U-(\theta+\gamma)}\frac{\lambda_V}{\lambda_V+(\theta+\gamma)} \\ &= \frac{\lambda_U}{\lambda_U-(\theta+\lambda_U-\lambda_V)}\frac{\lambda_V}{\lambda_V+(\theta+\lambda_U-\lambda_V)} \\ &= \frac{\lambda_V}{\lambda_V-\theta}\frac{\lambda_U}{\lambda_U+\theta}\,. \end{aligned}$$

Therefore, we see that the

$$X_i = U_i - V_i$$

variables under the exponential tilted (by $\gamma = \lambda_U - \lambda_V$) distribution have the component distributions

$$U \sim \text{Exponential}(\lambda_V) \text{ and } V \sim \text{Exponential}(\lambda_U)$$

instead of the original configurations.

[14]:
```
%%time

rng = rnd.default_rng(1)

γ = λ_U - λ_V
giveUpTime = 10**3
R = 10**6

alive = np.full(R, True)
S_n = np.zeros(R)
LRs = np.ones(R)

for n in range(1, giveUpTime+1):
    # Simulate the running sum from
    # the IS proposal distribution
    U_n = rng.exponential(1/λ_V, R)
    V_n = rng.exponential(1/λ_U, R)
    X_n = U_n - V_n
    S_n += X_n

    # Find the ones which go bankrupt after
    # this n-th claim has arrived.
    bankruptNow = (S_n > u) & alive

    # Store the likelihood ratio of this
    # simulation.
    LRs[bankruptNow] = np.exp(-γ*S_n[bankruptNow])

    # Record that this simulation is no
    # longer running.
    alive[bankruptNow] = 0

    # Quit after all R simulations have hit
    # bankruptcy.
    if np.sum(alive) == 0:
        break

if n == giveUpTime:
    print("We need to keep simulating!")

ests = LRs
ellHat = ests.mean()
sigmaHat = ests.std()
```

```python
widthCI = 1.96 * sigmaHat / np.sqrt(R)

print(f"CMC estimate:\t {ellHat} (+/- {widthCI})")
print(f"CMC low bound:\t {ellHat-widthCI}")
print(f"CMC upp bound:\t {ellHat+widthCI}")
```

```
CMC estimate:    4.060000097824725e-06 (+/- 1.3874567384834677e-07)
CMC low bound:   3.921255304398903e-06
CMC upp bound:   4.1987466520955965e-06
Wall time: 199 ms
```