Rare-event simulation: Code demo PyMC3

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March 26, 2020

[1]:
```
import numpy as np
import pandas as pd
import pymc3 as pm

%config InlineBackend.figure_format = 'retina'
import matplotlib.pyplot as plt
import seaborn as sns
sns.set()
```

[2]:
```
import sys
print("Python version:", sys.version)
print("Numpy version:", np.__version__)
print("PyMC3 version:", pm.__version__)
```

```
Python version: 3.7.7 (default, Mar 23 2020, 23:19:08) [MSC v.1916 64 bit
(AMD64)]
Numpy version: 1.18.1
PyMC3 version: 3.8
```

[3]:
```
df = pd.read_csv("intervals.csv")
```

[4]:
```
df.head()
```

[4]:

	EL	ER	SL	SR
0	0	45.999306	49.0	49.999306
1	14	43.999306	44.0	54.999306
2	39	49.375000	53.0	53.500000
3	0	35.999306	35.0	35.999306
4	0	43.999306	0.0	43.999306

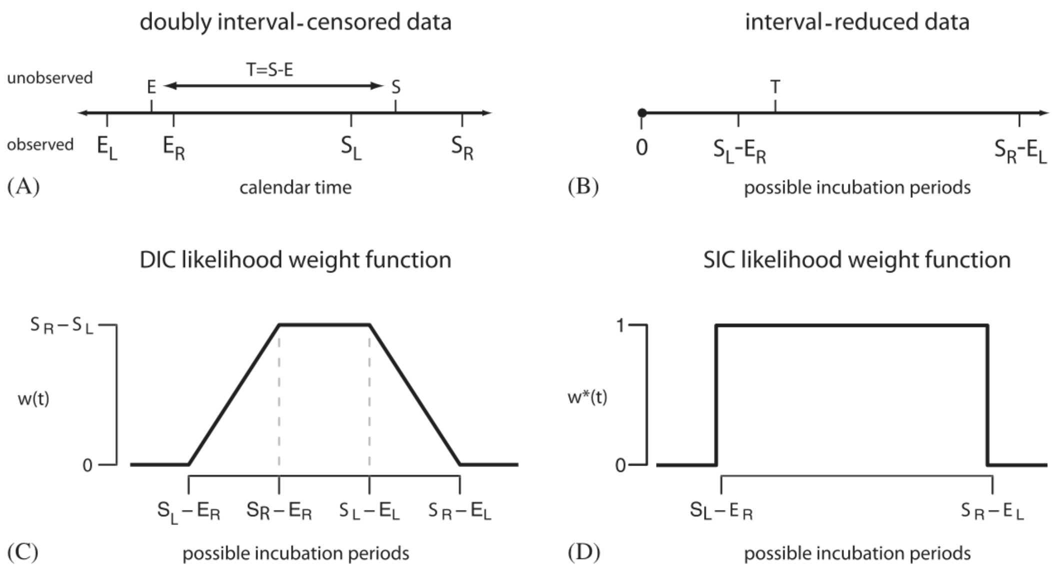
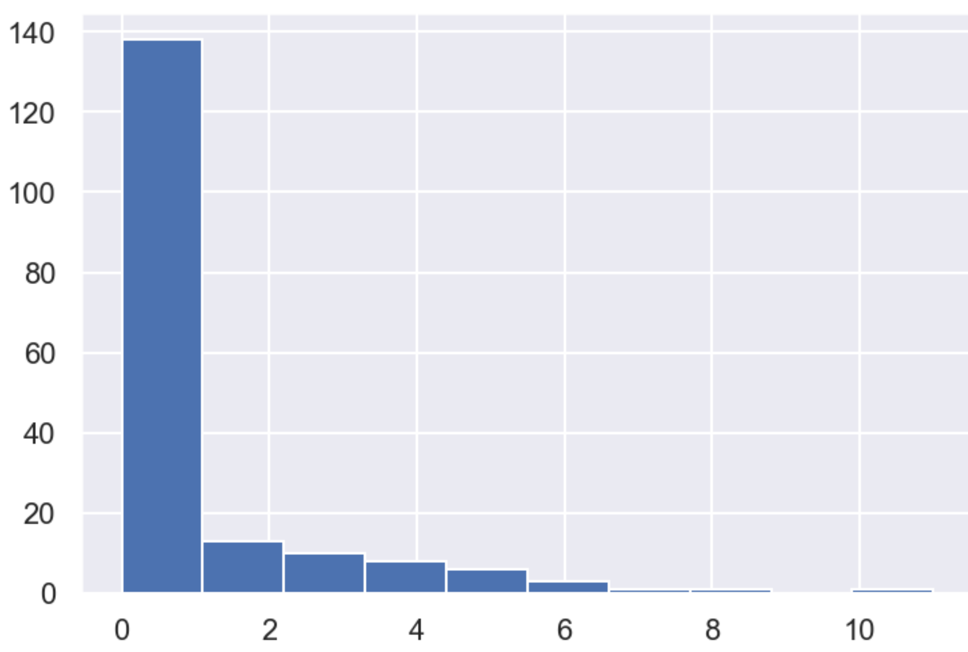


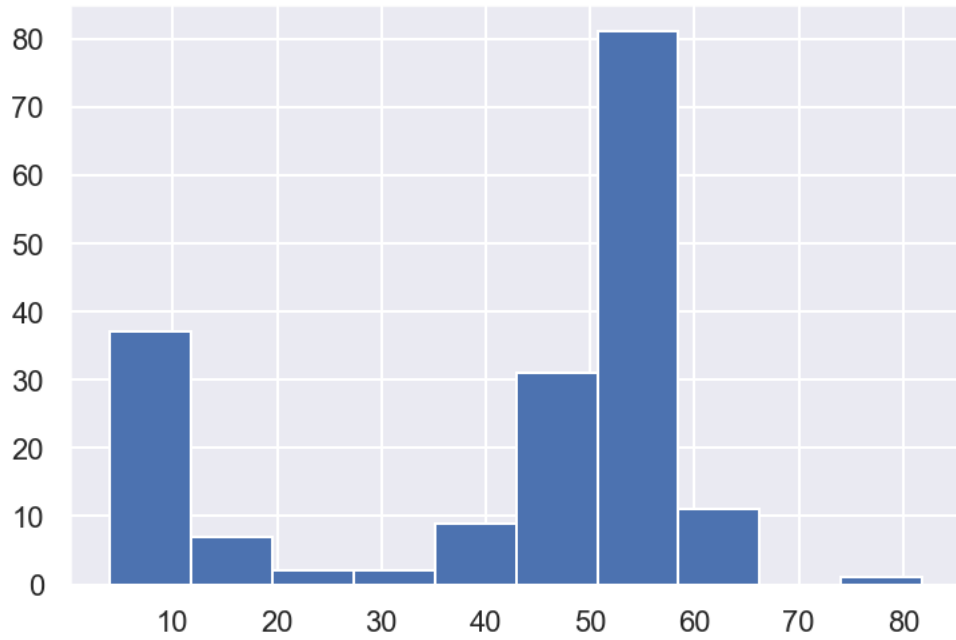
Figure 1 from Reich et al. (2009), *Estimating incubation period distributions with coarse data.*

[5]:
```
Tmin = np.array(np.maximum(df["SL"]-df["ER"], 0))
Tmax = np.array(df["SR"]-df["EL"])
```

[6]:
```
plt.hist(Tmin);
```



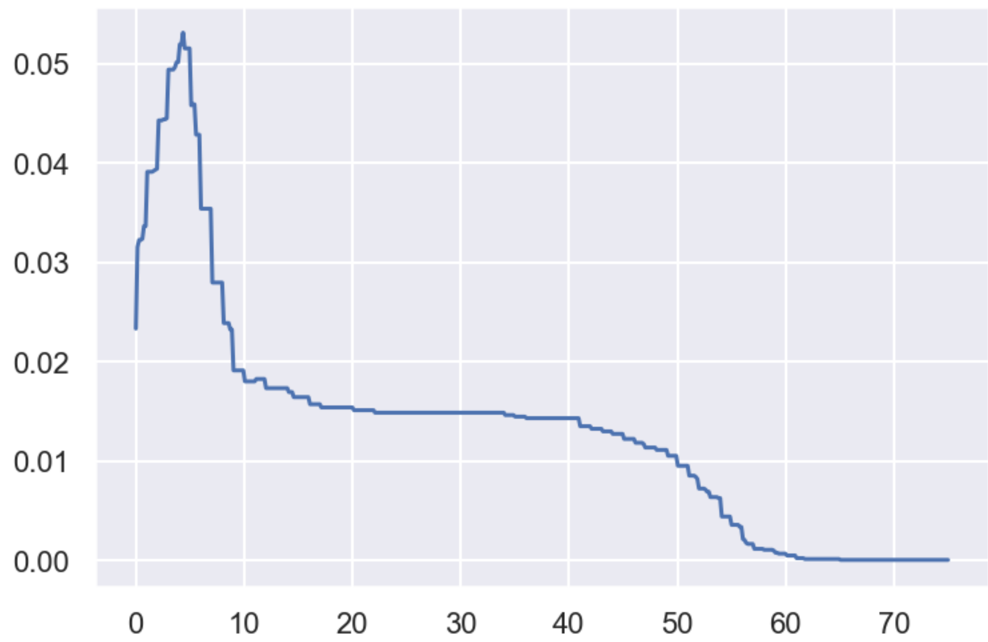
[7]:
```
plt.hist(Tmax);
```



[8]:
```
xs = np.linspace(0, 75, 500)
ys = np.zeros(len(xs))

nCases = len(Tmin)
for i in range(nCases):
    ys[(xs >= Tmin[i]) & (xs <= Tmax[i])] += 1 / (nCases *
 ↪(Tmax[i]-Tmin[i]))

plt.plot(xs, ys);
```



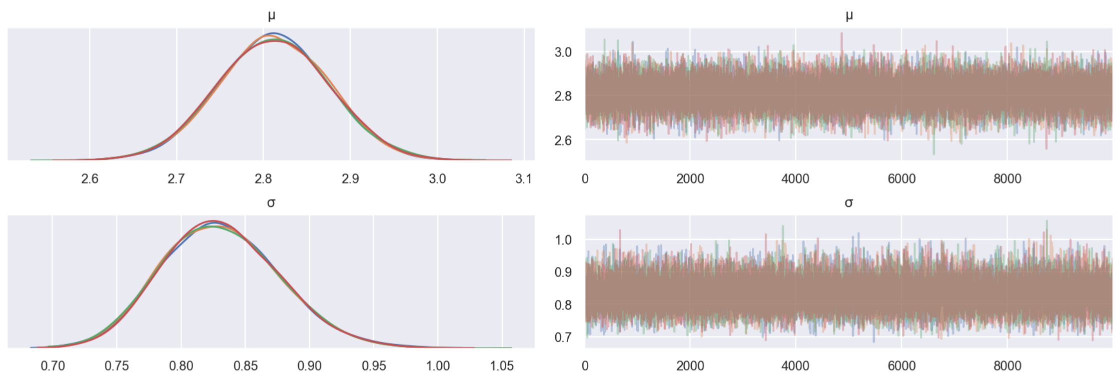

```
[9]: Tavgs = (Tmin + Tmax) / 2
```

```
[10]: %%time
with pm.Model() as model:
    μ = pm.Uniform('μ', lower=-25, upper=25)
    σ = pm.Uniform('σ', lower=0, upper=25)
    T = pm.Lognormal('T', mu=μ, sigma=σ, observed=Tavgs)
    trace = pm.sample(10**4)
```

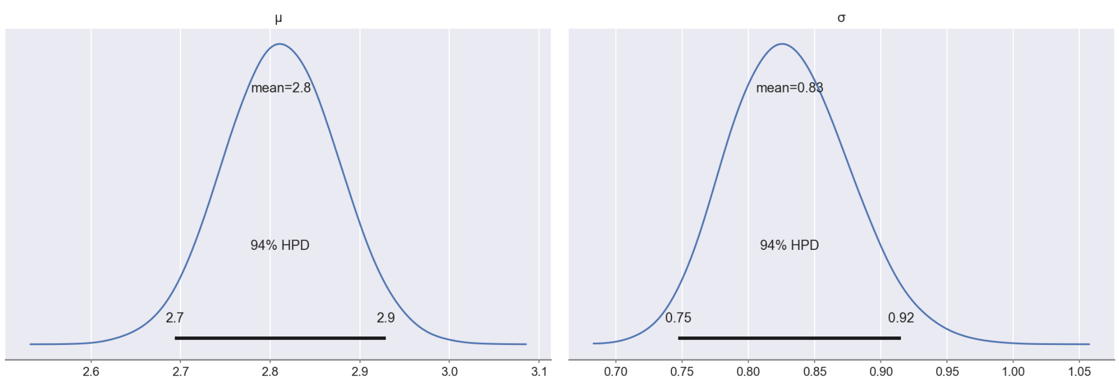
```
Auto-assigning NUTS sampler…
Initializing NUTS using jitter+adapt_diag…
Multiprocess sampling (4 chains in 4 jobs)
NUTS: [σ, μ]
Sampling 4 chains, 0 divergences:
100%|██████████████████████████████████████████| 42000/42000
[00:19<00:00, 2124.38draws/s]

Wall time: 53 s
```

```
[11]: pm.plot_trace(trace);
```




```
[12]: pm.plot_posterior(trace);
```



This is assuming we have observations $\boldsymbol{T} = (T_1, \ldots, T_n)$ which gives us a likelihood of

$$L(\mu, \sigma \mid \boldsymbol{T}) = \prod_{i=1}^{n} f(T_i; \mu, \sigma)$$

where $f(x; \mu, \sigma)$ is the p.d.f. of the $\mathsf{LogNormal}(\mu, \sigma^2)$ distribution.

However we don't have observations, just intervals. Say each unobserved period fell into $T_i \in [T_i^-, T_i^+]$. Our likelihood becomes

$$L(\mu, \sigma \mid \boldsymbol{T}^-, \boldsymbol{T}^+) = \prod_{i=1}^{n} \left[F(T_i^+; \mu, \sigma) - F(T_i^-; \mu, \sigma) \right]$$

where $F(x; \mu, \sigma)$ is the c.d.f. of the $\mathsf{LogNormal}(\mu, \sigma^2)$ distribution.

```
[13]: import theano.tensor as tt

# Taken from PyMC3's pymc3/distributions/dist_math.py file
```

```
# starting at line 346.
def zvalue(x, sigma, mu):
    """
    Calculate the z-value for a normal distribution.
    """
    return (x - mu) / sigma

# Taken from PyMC3's pymc3/distributions/continuous.py file
# starting at line 1849.
def cdf(x, mu, sigma):
        """
        Compute the log of the cumulative distribution function for
 ↪Lognormal distribution
        at the specified value.

        Parameters
        ----------
        x: numeric
            Value(s) for which log CDF is calculated. If the log CDF for
 ↪multiple
            values are desired the values must be provided in a numpy array
 ↪or theano tensor.

        Returns
        -------
        TensorVariable
        """
        z = zvalue(np.log(x), mu=mu, sigma=sigma)

        return tt.switch(
                tt.lt(z, -1.0),
                tt.erfcx(-z / tt.sqrt(2.)) / 2. * np.exp(-tt.sqr(z) / 2),
                tt.erfc(-z / tt.sqrt(2.)) / 2.
        )
```

With `Potential` we have to add log-terms to the likelihood. So

$$\log\left[L(\mu,\sigma \mid \boldsymbol{T}^-,\boldsymbol{T}^+)\right] = \sum_{i=1}^{n} \log\left[F(T_i^+;\mu,\sigma) - F(T_i^-;\mu,\sigma)\right].$$

[14]:
```
%%time
with pm.Model() as model:
    μ = pm.Uniform('μ', lower=-25, upper=25)
    σ = pm.Uniform('σ', lower=0, upper=25)
    pm.Potential('T', tt.sum(tt.log( cdf(Tmax, μ, σ) - cdf(Tmin, μ, σ) )))
    trace = pm.sample(10**5, step=pm.Metropolis())
```

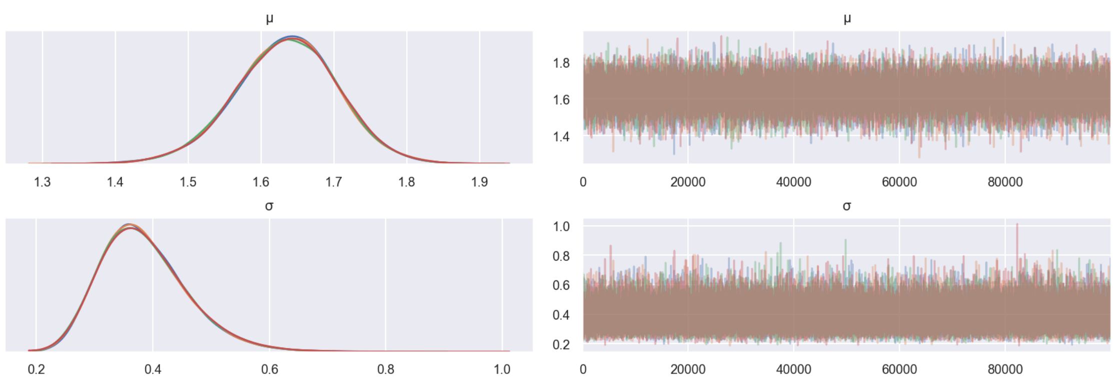
```
Multiprocess sampling (4 chains in 4 jobs)
CompoundStep
>Metropolis: [σ]
>Metropolis: [μ]
Sampling 4 chains, 0 divergences:
100%|██████████████████████████████████████████████| 402000/402000
[03:16<00:00, 2040.71draws/s]
The number of effective samples is smaller than 10% for some parameters.

Wall time: 3min 41s
```

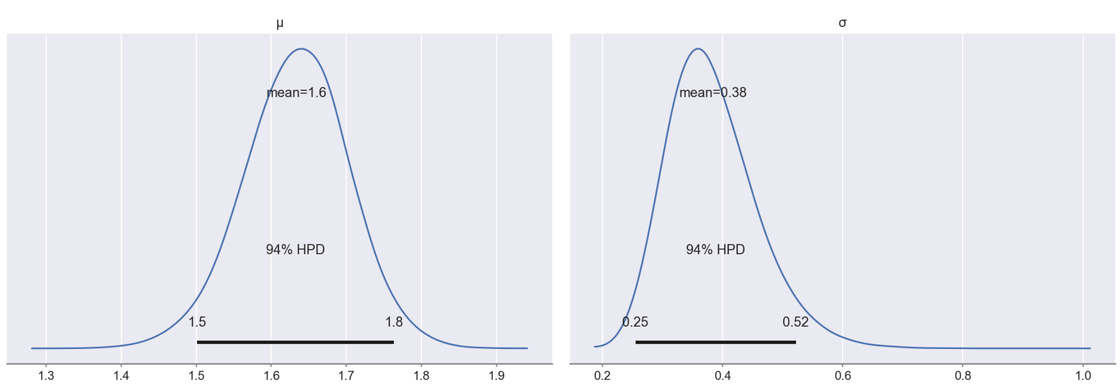
[15]:
```python
pm.stats.ess(trace["μ"]), pm.stats.ess(trace["σ"])
```

[15]: (30595.937405029406, 55537.902861184826)

[16]:
```python
pm.plot_trace(trace);
```



[17]:
```python
pm.plot_posterior(trace);
```



[18]:
```python
trace["μ"].mean()
```

[18]: 1.6333400315715902

[19]:
```
trace["σ"].mean()
```

[19]: 0.3841109415517976